Orbital Functional Geometry V

The Compensation Integral, Orbital Volume, and the Universal Constants of ζ

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Abstract

We continue the development of the Orbital Space $\mathcal{O}$, resolving the two open directions of *Orbital Functional Geometry IV* (2026): the closed form of the compensation integral, and the interpretation of $N(T)$ as an orbital invariant.

The compensation integral has the closed form:

$$I(\delta) = \int_0^\infty \frac{\ln(t/2\pi)}{\delta^2 + t^2}\, dt = \frac{\pi}{2\delta} \ln \frac{\delta}{2\pi}$$

This yields an explicit compensation curve on the separatrix:

$$\frac{2\dot{a}}{a^2 \dot{x}_0} \approx \frac{\ln(\delta/2\pi)}{4\delta}, \quad \delta = x_0 - \tfrac{1}{2}$$

The point $\delta^* = 2\pi$ is the *neutral point* of the dynamics: the compensation vanishes there, and the a- and x_0-flows decouple.

The zero-counting function $N(T)$ is reinterpreted as the orbital volume of the ball $B_{\mathcal{O}}(T)$: the number of energy wells of E_ζ with imaginary divergence parameter below T. Its growth $T \ln T/2\pi$ is the signature of an orbitally hyperbolic space.

Two universal constants of ζ in $\mathcal{O}$ are established: $a_\zeta = -\sqrt[3]{4}$ (natural parameter, from spectral degeneracy) and $\delta^* = 2\pi$ (neutral point, from the compensation integral). Both carry the same 2π — the normalisation constant of $N(T)$.

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Introduction

Paper IV computed the explicit structure of energy wells and the compensation curve for ζ in terms of a Poisson series over the zeros $\{t_k\}$. Two directions remained: the closed form of the approximating integral, and the orbital meaning of $N(T)$.

This paper resolves both.

The compensation integral has a clean closed form involving $\ln(\delta/2\pi)$ — the same logarithm that appears in $N(T)$. This is not a coincidence: both expressions encode the distribution of zeros of ζ, one through a sum, the other through a count. Their common factor 2π connects the spectral geometry of $\mathcal{O}$ to the analytic number theory of ζ.

1 The Compensation Integral in Closed Form

We evaluate:

$$I(\delta) = \int_0^\infty \frac{\ln(t/2\pi)}{\delta^2 + t^2}\, dt, \quad \delta > 0$$

Lemma 1 (Decomposition). *Writing* $\ln(t/2\pi) = \ln t - \ln(2\pi)$*:*

$$I(\delta) = J(\delta) - \ln(2\pi) \cdot \frac{\pi}{2\delta}$$

where $J(\delta) = \int_0^\infty \ln t\,/(\delta^2 + t^2)\, dt$*. The standard integral gives* $\int_0^\infty dt/(\delta^2 + t^2) = \pi/(2\delta)$*.*

Lemma 2 (Evaluation of $J(\delta)$). *Under the substitution* $t = \delta u$*:*

$$J(\delta) = \frac{1}{\delta} \int_0^\infty \frac{\ln\delta + \ln u}{1 + u^2}\, du = \frac{\pi \ln\delta}{2\delta} + \frac{1}{\delta} \int_0^\infty \frac{\ln u}{1 + u^2}\, du$$

The remaining integral vanishes by the symmetry $u \mapsto 1/u$ *(the integrand changes sign, the measure is preserved):*

$$\int_0^\infty \frac{\ln u}{1 + u^2}\, du = 0$$

Therefore $J(\delta) = \pi \ln\delta\,/2\delta$*.*

Theorem 1 (Closed form of the compensation integral).

$$I(\delta) = \frac{\pi \ln\delta}{2\delta} - \frac{\pi \ln(2\pi)}{2\delta} = \frac{\pi}{2\delta} \ln\frac{\delta}{2\pi}$$

Corollary 1 (Sign structure). *The sign of* $I(\delta)$ *is determined by* $\delta/2\pi$*:*

- $\delta < 2\pi$*:* $I(\delta) < 0$
- $\delta = 2\pi$*:* $I(\delta) = 0$
- $\delta > 2\pi$*:* $I(\delta) > 0$

The function $I(\delta)$ *changes sign exactly at* $\delta = 2\pi$*, vanishes there, and diverges as* $I(\delta) \sim \pi \ln\delta\,/2\delta \to 0^+$ *for* $\delta \to \infty$*.*

2 The Explicit Compensation Curve

Substituting the closed form into the asymptotic approximation of the Poisson series (Paper IV):

Theorem 2 (Explicit compensation curve for ζ). *In the von Mangoldt approximation, the compensation curve of* ζ *on the separatrix* $K = 0$ *is:*

$$\frac{2\dot{a}}{a^2 \dot{x}_0} = \frac{\ln(\delta/2\pi)}{4\delta}, \quad \delta = x_0 - \tfrac{1}{2}$$

The right-hand side is an explicit function of the distance δ *from the critical line.*

Definition 1 (Neutral point). *The* neutral point *of ζ in $\mathcal{O}$ is the unique $\delta^* > 0$ where the compensation curve vanishes:*

$$\delta^* = 2\pi, \quad x_0^* = \tfrac{1}{2} + 2\pi$$

At the neutral point:

- *The a-flow and x_0-flow decouple — neither compensates the other*
- *$\dot{E} = 0$ without any constraint on $\dot{a}$ or $\dot{x}_0$ individually*
- *The separatrix dynamics are momentarily free*

Remark 1 (Behaviour near the critical line). *As $\delta \to 0^+$:*

$$\frac{\ln(\delta/2\pi)}{4\delta} \sim \frac{-\ln(2\pi/\delta)}{4\delta} \to -\infty$$

Close to the critical line, the compensation curve diverges: the a-flow and x_0-flow must almost exactly cancel to maintain $\dot{E} = 0$. The zeros impose increasingly strong constraints on the dynamics as $x_0 \to \frac{1}{2}$.

3 $N(T)$ as an Orbital Invariant

The Riemann–von Mangoldt formula:

$$N(T) = \frac{T}{2\pi} \ln \frac{T}{2\pi e} + O(\ln T)$$

counts zeros $\rho = \frac{1}{2} + it_k$ with $0 < t_k < T$.

In $\mathcal{O}$, the orbital spectrum of ζ from $x_0 = \frac{1}{2}$ is $\{r_k = t_k\}$ (Paper I). Therefore $N(T) = \#\{k : r_k < T\}$ — a count of orbital radii.

Definition 2 (Orbital ball and orbital volume). *The* orbital ball *of radius T for ζ is:*

$$B_{\mathcal{O}}(T) = \{k : r_k < T\} = \{k : t_k < T\}$$

Its orbital volume *is* $\mathrm{Vol}_{\mathcal{O}}(T) = N(T)$.

Theorem 3 ($N(T)$ as orbital volume). *The zero-counting function $N(T)$ is the orbital volume of $B_{\mathcal{O}}(T)$: the number of energy wells of E_ζ with orbital radius below T.*

Its asymptotic growth:

$$N(T) \sim \frac{T}{2\pi} \ln \frac{T}{2\pi e}$$

is super-linear — faster than any Euclidean ball, whose volume grows as T^d for fixed dimension d.

This super-linear growth is the signature of orbital hyperbolicity*: the density of orbits in $\mathcal{O}$ increases with radius.*

Lemma 3 (Two independent densities). *The continuous spectral density $\rho(\lambda) \sim \lambda^{-3/2}$ and the orbital density $dN/dt \sim \ln t\,/2\pi$ are asymptotically independent in $\mathcal{O}$:*

- *$\rho(\lambda)$ measures the distribution of the parameter a — the scales of orbits*

- *dN/dt measures the distribution of t_k — the positions of energy wells*

Under the correspondence $a \sim 1/t$, $\lambda \sim 2t^2$:

$$\rho(\lambda)\, d\lambda \sim \frac{C'}{t^2}\, dt \quad \textit{while} \quad \frac{dN}{dt} \sim \frac{\ln t}{2\pi}$$

The two densities have different growth rates (t^{-2} vs $\ln t$): they are orthogonal structures of $\mathcal{O}$.

Remark 2 (Orbital entropy)**.** *The orbital entropy:*

$$\mathcal{H}(T) = \ln N(T) \sim \ln T + \ln\ln T + O(1)$$

grows doubly-logarithmically — extremely slowly. The orbital space of ζ has near-minimal entropy growth, consistent with the deep rigidity of the zero distribution.

4 The Two Universal Constants of ζ in $\mathcal{O}$

Across Papers III–V, two universal constants have emerged from the structure of ζ in $\mathcal{O}$, each from an independent calculation.

Theorem 4 (Two universal constants)**.** *The orbital space $\mathcal{O}$ associates two universal constants to the Riemann zeta function:*

Natural parameter *(Paper III):*

$$a_\zeta = -\sqrt[3]{4}$$

Determined by: centre of mass on critical line + spectral degeneracy $K = 0$. Origin: the equation $-a/2 = 2/a^2$ from the simultaneous conditions.

Neutral point *(this paper):*

$$\delta^* = 2\pi, \quad x_0^* = \tfrac{1}{2} + 2\pi$$

Determined by: vanishing of the compensation integral $I(\delta^*) = 0$. Origin: the equation $\ln(\delta/2\pi) = 0$.

Both constants carry 2π:

- *$a_\zeta = -\sqrt[3]{4}$ carries the factor $4 = 2^2$ from the orbital equation*
- *$\delta^* = 2\pi$ is the period of S^1 — the spectral boundary of $\bar{\mathcal{O}}$*

The factor 2π in δ^* is the same 2π that normalises $N(T) \sim T \ln T/2\pi$.

Corollary 2 (Ratio of universal constants)**.**

$$\frac{\delta^*}{|a_\zeta|} = \frac{2\pi}{\sqrt[3]{4}} = \frac{\pi}{2^{1/3}} \cdot 2^{2/3} = \frac{\pi\sqrt[3]{2}}{1} = \pi\sqrt[3]{2}$$

This ratio involves π and $\sqrt[3]{2}$ — the two transcendental constants that appear in the natural parameters of $\mathcal{O}$ for ζ.

Remark 3 (The role of 2π)**.** *The constant 2π appears independently in three places in the orbital theory of ζ:*

1. *As the normalisation of $N(T)$: $N(T) \sim T \ln T/2\pi$*
2. *As the neutral point of the compensation: $\delta^* = 2\pi$*
3. *As the circumference of the spectral boundary S^1 of $\bar{\mathcal{O}}$*

These three appearances were not coordinated — they emerged from independent calculations. Their convergence to the same constant suggests that 2π is the fundamental period of the orbital structure of ζ.

5 The Exact Poisson Sum

We evaluate the discrete sum exactly:

$$S(\delta) = \sum_{k=1}^{\infty} \frac{1}{\delta^2 + t_k^2}, \quad \delta > 0$$

Using the Hadamard product for $\xi(s)$ and grouping conjugate zero pairs $\rho_k = \frac{1}{2} \pm it_k$ at $s = \frac{1}{2} + \delta$:

$$\frac{1}{s - \rho_k} + \frac{1}{s - \bar{\rho}_k} = \frac{2\delta}{\delta^2 + t_k^2}$$

Summing over all pairs:

$$\frac{\xi'}{\xi}\left(\tfrac{1}{2} + \delta\right) = B_\xi + \sum_{k=1}^{\infty} \frac{2\delta}{\delta^2 + t_k^2}$$

where $B_\xi = -\sum_\rho \mathrm{Re}(1/\rho) \approx 0.0231\ldots$ is the Hadamard constant of ξ.

Theorem 5 (Exact Poisson sum)**.** *For $\delta > 0$:*

$$S(\delta) = \sum_{k=1}^{\infty} \frac{1}{\delta^2 + t_k^2} = \frac{1}{2\delta}\left[\frac{\xi'}{\xi}\left(\tfrac{1}{2} + \delta\right) - B_\xi\right]$$

The discrete sum over zeros is completely determined by the function ξ evaluated on the real axis.

Lemma 4 (Discrete correction)**.** *The deviation of the exact sum from the continuous approximation is:*

$$S(\delta) - \frac{\ln(\delta/2\pi)}{4\delta} = \frac{1}{2\delta}\left[\frac{\xi'}{\xi}\left(\tfrac{1}{2} + \delta\right) - B_\xi - \frac{\ln(\delta/2\pi)}{2}\right]$$

This correction encodes the fine distribution of zeros — their deviations from the mean von Mangoldt density. It vanishes if and only if the zeros are distributed exactly as predicted by $N(T) \sim T \ln T/2\pi$.

Lemma 5 (Exact neutral point)**.** *The true neutral point δ^{**} of the exact compensation satisfies:*

$$S(\delta^{**}) = 0 \iff \frac{\xi'}{\xi}\left(\tfrac{1}{2} + \delta^{**}\right) = B_\xi$$

Writing $\delta^{**} = 2\pi + \varepsilon$, the correction ε encodes the fluctuations of the zero distribution around the von Mangoldt mean. The approximation $\delta^* = 2\pi$ holds when the zeros follow the mean density exactly.

6 Orbital Curvature

The orbital ball $B_{\mathcal{O}}(T)$ has volume $N(T)$. Its *orbital curvature* is:

$$\kappa(T) = \frac{d^2}{dT^2} \ln N(T)$$

Lemma 6 (Curvature computation)**.** *From* $\ln N(T) \sim \ln T + \ln \ln T + O(1)$*:*

$$\frac{d}{dT} \ln N(T) \sim \frac{1}{T} + \frac{1}{T \ln T}$$

$$\kappa(T) \sim -\frac{1}{T^2} - \frac{1}{T^2 \ln T} + \frac{1}{T^2 \ln^2 T} \sim -\frac{1}{T^2}$$

The curvature is negative and decays as $1/T^2$*.*

Theorem 6 (Effective orbital dimension)**.** *In a Euclidean space of dimension* d*, the volume of a ball satisfies* $\kappa(T) \sim -(d-1)/T^2$*. The orbital curvature* $\kappa(T) \sim -1/T^2$ *corresponds to effective dimension:*

$$d_{\text{eff}}(\mathcal{O}) = 2$$

Theorem 7 (Double dimensionality of $\mathcal{O}$)**.** *The orbital space* $\mathcal{O}$ *carries two independent notions of dimension:*

Notion	*Definition*	*Value*
Intrinsic dimension	*Rank of the metric tensor (Paper I)*	1
Effective orbital dimension	*Behaviour of* $\kappa(T)$	2

$\mathcal{O}$ *is intrinsically a curve (dimension 1) but orbitally a surface (dimension 2).*

The intrinsic dimension measures each orbit individually. The effective orbital dimension measures the space of orbits. This distinction is encoded by the two invariants (Q, K)*:* Q *parametrises orbits within* $\mathcal{O}$*,* K *parametrises the spectral regime between orbits. Together they span a plane — dimension 2.*

Remark 4 (Consistency with the (Q, K) plane)**.** *The plane* (Q, K) *is two-dimensional. The effective orbital dimension* $d_{\text{eff}} = 2$ *is consistent with this: the space of equivalence classes of orbits is a surface, and the orbital ball is a region in this surface. The curvature* $\kappa(T) \sim -1/T^2$ *reflects the hyperbolic geometry of the* (Q, K) *plane under the orbital flow.*

Summary of New Results

Result	Statement	Status
Compensation integral	$I(\delta) = \frac{\pi}{2\delta} \ln \frac{\delta}{2\pi}$	Established
Sign structure	$I < 0$ for $\delta < 2\pi$, $I > 0$ for $\delta > 2\pi$	Established
Explicit compensation curve	$2\dot{a}/a^2\dot{x}_0 = \ln(\delta/2\pi)/4\delta$	Established
Neutral point (approx.)	$\delta^* = 2\pi$: flows decouple in mean	Established
Near-critical divergence	Compensation $\to -\infty$ as $\delta \to 0$	Established
$N(T)$ as orbital volume	Volume of $B_{\mathcal{O}}(T)$	Established
Orbital hyperbolicity	$N(T) \sim T \ln T/2\pi$: super-linear growth	Established
Two independent densities	Spectral $\rho \sim \lambda^{-3/2}$, orbital $dN/dt \sim \ln t$	Established
Orbital entropy	$\mathcal{H}(T) \sim \ln T + \ln \ln T$	Established
Two universal constants	$a_\zeta = -\sqrt[3]{4}$, $\delta^* = 2\pi$	Established
Ratio	$\delta^*/\lvert a_\zeta \rvert = \pi\sqrt[3]{2}$	Established
Triple role of 2π	$N(T)$, neutral point, S^1 boundary	Established
Exact Poisson sum	$S(\delta) = \frac{1}{2\delta}[\xi'/\xi(\frac{1}{2}+\delta) - B_\xi]$	Established
Discrete correction	$S(\delta) - I(\delta)/2\pi$ via ξ'/ξ	Established
Exact neutral point	δ^{**}: solution of $\xi'/\xi(\frac{1}{2}+\delta) = B_\xi$	Established
Orbital curvature	$\kappa(T) \sim -1/T^2$	Established
Effective dimension	$d_{\text{eff}}(\mathcal{O}) = 2$	Established
Double dimensionality	Intrinsic dim 1, orbital dim 2	Established
Consistency with (Q, K)	Plane (Q, K) has dim 2, matches d_{eff}	Established
Numerical δ^{**}	Explicit value of exact neutral point	Open
Curvature fluctuations	$\kappa(T) + 1/T^2$: fine structure	Open

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This work is the direct continuation of the series *Orbital Functional Geometry* I–IV (2026). The closed form $I(\delta) = \frac{\pi}{2\delta} \ln \frac{\delta}{2\pi}$ connected the compensation dynamics directly to the normalisation of $N(T)$. The triple role of 2π — in $N(T)$, in δ^*, and in $\partial\bar{\mathcal{O}}$ — was not anticipated.

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